UNIVERSITY OF PENNSYLVANIA

ELECTRONIC CONDUCTION AND IONIZATION IN CROSSED ELECTRIC AND MAGNETIC FIELDS

NICHOLAS MINORSKY

A THESIS

IN PHYSICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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ELECTRONIC CONDUCTION AND IONIZATION IN CROSSED ELECTRIC AND MAGNETIC FIELDS.

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INTRODUCTION.

The action of a magnetic field on a discharge has been the object of many investigators. Comparatively simple is the case when the effect occurs in vacuo (i.e., pure electron discharge).[1, 2]

When gas is admitted into such a tube, the effect of the magnetic field becomes less marked, as if the gas had a tentency to maintain the current in spite of the action of the magnetic field.[3]

When the ionization of the gas sets in, the conditions become too complicated for a theoretical analysis to be attempted; there exists considerable literature relative to the effect of magnetic field on the discharge, especially in connection with the phenomena of rotation of discharges in radial fields.[4] When there is no ionization in the gas it seems possible to extend the original theory of Sir. J. J. Thomson in combination with the mechanism of elastic collisions between the primary electrons and the gas particles so as to explain the gradual drift of the electrons through the gas. This part

[1] J. J. Thomson, "Conduction of Electricity through Gases," Chapter IV, 1906.

[2] A. W. Hull, *Physical Review*, Vol. 18, 1921.

[3] O. W. Richardson and R. Chaudhuri, *Phil. Mag.* (6), **45**, 1923.

[4] A. de la Rive, *Arch. Se. phys. nat.*, t. 12 (1848), p. 222; *C. R.* (1849), p. 414; *Pogg. Ann.* (1858), p. 129; *Phil. Mag.* (4), t. 33 (1867), p. 512; H. A. Wilson and Martin, *Proc. Roy. Soc.*, t. 79 (1900), p. 417; D. N. Mallik, *Phil. Mag.* (6), t. 16 (1908), p. 531; A. Dufour, *Ann. de Chim. and Phys.* (8), t. 22 (1911), p. 282; C. E. Guye and A. Rothen, *Arch. Sc. Phys. nat.* (5), t. 3, p. 441; J. Nicol, *Proc. Roy. Soc.*, *A*, t. 82 (1909), p. 29; H. Stolt, *Ann. der Phys.* (4), t. 74 (1924), p. 80; A. Hagenbach, "Handbuch der Physik" (1927), t. 14, Chap. VI, p. 337; N. Minorsky, *Journal de Physique*, April 1928.

of the problem is comparatively simple and is treated in the second section of this paper.

It is more difficult to obtain an expression for space charge conditions even in the simplest case of cylindrical symmetry which is assumed in the following. There seem to be two distinct methods of attack. The first method consists of introducing an indeterminate function representing the electronic density at a certain level, and of formulating the conditions of steady state at this level—this leads to an integral equation of the Fredholm type with a kernel representing the probability of transition between the electronic trajectories.[5] The difficulty of this method lies in the fact that the kernel of the equation is not known, since it depends upon the unknown mechanism of collisions between electrons and the gas particles. Its chief interest lies in the fact that it establishes the existence of a rigorous analytical solution on the basis of the classical theory.

The second method, developed in Section 3, is based on the assumption of a certain simplified mechanism of scattering in which instead of the double infinity of trajectories generated by collisions, one single trajectory is considered. To this average trajectory is attached a certain statistical weight resulting from the geometry of trochoidal motions. This imaginary distribution by discreet trajectories is, of course, very different from the actual distribution on a scale of the order of magnitude of trochoidal loops.[6] As regards the macroscopic scale (i.e., dimensions of the apparatus) presumably the change is very slight and it approaches zero when the size of the loops decreases.

Analytically the approximation consists of substituting for an integral equation a system of a finite number of linear equations with a suitable parameter.[7] Under those conditions it is possible to apply the binomial theorem of the theory of discontinuous probabilities and to gain thus an approximate

[5] This will form the subject of a later publication.

[6] In case of low voltage discharges (e.g., hot cathode tubes) and magnetic fields of the order 500–1000 gauss, the order of magnitude of trochoidal loops is roughly 10^{-2}–10^{-3} cm.

[7] R. Courant and D. Hilbert, "Methoden der Mathematischen Physik," Chapters I, II.

insight into the mechanism of the electronic propagation through an inert gas when the magnetic field is acting at a right angle to the electric force, and also to formulate the space charge conditions.

It must be noted that the mechanism of electronic conduction through the gas in crossed fields is essentially different from the case when the magnetic field is absent, since in the former the electrons move on the average at right angles to the electric force, whereas in the latter they move in its direction. One can also see this from the fact that the two phenomena do not tend to the same limit when the mean free path in the gas increases indefinitely. In fact, in vacuo, the electronic conduction, when only the electric force is present, is governed by the Langmuir's 3/2 power law, whereas with crossed fields it disappears entirely for certain critical values of the field.[2]

In cases where the size of trochoidal loops becomes considerable in comparison with the mean free path, the effect of magnetic field is less marked and the space charge effect caused by magnetic field disappears gradually for decreasing H.

In order to emphasize the characteristic feature of the phenomenon it will be assumed in the following that the length of one trochoidal loop is small in comparison with the mean free path.

GEOMETRY OF THE TROCHOIDAL SCATTERING OF ELECTRONS IN CROSSED FIELDS.

It is useful to simplify the theoretical investigation by introducing a number of assumptions under which the fundamental effect of scattering is particularly simple.[8]

It is always possible to waive one by one these assumptions and to study the more general cases. These assumptions are:

1. The gas is monatomic and without any appreciable electron affinity (e.g., noble gases, metallic vapors).
2. Collisions are elastic.
3. Velocities of thermal agitation are negligible.

[8] For the sake of simplicity the arrangement shown on Fig. 1 of the writer's article, published in *Journal de Physique* (Apr. 1928) will be assumed, with the exception that a circular filament is substituted for the pool of mercury. The emission in such a case occurs from each point of the abcissa axis.

4. The space charge effect is negligible (electric force is constant).
5. Initial velocities of the electrons are negligible.
6. The magnetic force is constant throughout the whole region.
7. The motion of the particles is studied in a plane tangent to the cylindrical surface (i.e., the radial motion of particles is neglected).

Of these assumptions, 1 and 2 can always be approximated sufficiently well by suitable experimental conditions; 3, 5, 6 and 7 are in general fulfilled sufficiently well. There remains only assumption 4, which generally is not fulfilled unless one works with very low magnetic fields and low electron emission. Unfortunately, there seems to be no method of proceeding initially without making this assumption, but the determination of the space charge constitutes the final object of the problem. The practical issue from this vicious circle is to admit assumption 4 and to proceed to determine the electron distribution to a first approximation; after the space charge conditions are thus determined it is necessary to start a second approximation in which for condition 4 there will be substituted the results from the first approximation.

The equations of motion of a charged particle in crossed fields are

$$\begin{aligned} m\frac{d^2x}{dt^2} &= He\frac{dy}{dt}, \\ m\frac{d^2y}{dt^2} &= -He\frac{dx}{dt} + eE. \end{aligned} \tag{1}$$

Putting $b = He/m$; $c = eE/m$ and transforming the variables x, y, into ξ, η, τ by means of the substitution

$$\begin{aligned} x &= b^{-2}c\xi, \\ y &= b^{-2}c\eta, \\ t &= b^{-1}\tau, \end{aligned} \tag{2}$$

equations (1) become:

$$\begin{aligned} \ddot{\xi} &= \dot{\eta} \\ \ddot{\eta} &= -\dot{\xi} + 1, \end{aligned} \tag{3}$$

where the dots designate the derivations with respect to τ.

The substitution (2) simplifies considerably the subsequent calculations; it means that instead of dealing with the lengths (x, y) measured in cm. and the time (t) measured in seconds, the new coördinates of space $\xi = x/b^{-2}c$; $\eta = y/b^{-2}c$ and time $\tau = tb$ are expressed in terms of $b^{-2}c = E/kH^2 = a$ (i.e., radius of rolling circle of the cycloid given by (1) and $1/b = 1/kH$ (i.e., period of the electron in its orbit).

The integration of equations (3) with the initial conditions

$$\xi = \eta = \dot{\xi} = \dot{\eta} = 0 \text{ when } \tau = 0 \text{ gives } \xi = \tau - \text{Sin } \tau \text{ and } \eta = 1 - \text{Cos } \tau. \quad (4)$$

Equations (4) represent a cycloid.

A collision between a free electron and a molecule, occuring on this trajectory is characterized by two independent parameters: τ_0 and ϕ. The former (τ_0) indicates the point of the trajectory at which the collision occurs; the latter (ϕ) gives the angle between the incident and the reflected velocities of the electron. Assume, for example, that the collision occurs at the apex $(\tau_0 = \pi)$ of the trajectory and that it is of the "end on" type $(\phi = \pi)$. For the sake of convenience let such a collision be designated as of the $(\pi\pi)$ type (where the first π means $\tau_0 = \pi$ and the second that $\phi = \pi$). A $(\pi\pi)$ collision gives rise to new initial conditions (taking a new origin of time $t = 0$ at the instant of collision) viz., when $\tau = 0, \xi = \pi, \eta = 2, \dot{\xi} = -2, \dot{\eta} = 0$, which give in conjunction with (3):

$$\xi = \pi - 3 \text{ Sin } \tau + \tau \text{ and } \eta = -3 \text{ Cos } \tau + 5. \quad (5)$$

The curve given by equations (5) is a trochoid having the same *rolling radius as the original cycloid (4) and a generating radius three times its rolling radius.* A following $\pi\pi$ collision on the trochoidal trajectory given by equation (5) gives rise to a third trochoidal trajectory having *the same rolling* radius and a generating radius equal to *five* times its rolling radius, and so on.

The following table contains data relative to the first four trajectories generated by the first three elastic $(\pi\pi)$ collisions. For the sake of convenience the variables x, y, t are used instead of ξ, η, τ in this table. The constants of integration are not determined since only the velocity distribution as a function of y is of interest here $(\omega = \kappa H)$.

	x	y	$\frac{dx}{dt}$	$\frac{dy}{dt}$
First trajectory	$\frac{E}{H}t - a \operatorname{Sin} \omega t$	$a - a \operatorname{Cos} \omega t$	$\frac{E}{H}(1 - \operatorname{Cos} \omega t)$	$\frac{E}{H} \operatorname{Sin} \omega t$
Second trajectory	$\frac{E}{H}t - 3a \operatorname{Sin} \omega t$	$5a - 3a \operatorname{Cos} \omega t$	$\frac{E}{H}(1 - 3 \operatorname{Cos} \omega t)$	$\frac{E}{H} 3 \operatorname{Sin} \omega t$
Third trajectory	$\frac{E}{H}t - 5a \operatorname{Sin} \omega t$	$13a - 5a \operatorname{Cos} \omega t$	$\frac{E}{H}(1 - 5 \operatorname{Cos} \omega t)$	$\frac{E}{H} 5 \operatorname{Sin} \omega t$
Fourth trajectory	$\frac{E}{H}t - 7a \operatorname{Sin} \omega t$	$25a - 7a \operatorname{Cos} \omega t$	$\frac{E}{H}(1 - 7 \operatorname{Cos} \omega t)$	$\frac{E}{H} 7 \operatorname{Sin} \omega t$

(6)

For the points $\omega t = \pi$, $dy/dt = 0$ and $dx/dt = E/H \times 2, 4, 6, 8 \ldots$ hence the kinetic energies of the electron passing through these points are in ratio $1:4:9:16:25:\ldots$ If, on the other hand, from (6) one forms the values for y for the same points $\omega t = \pi$, one finds that these values are also in the same ratio: $1:4:9:16:25:\ldots$ Thus the kinetic energy increases in proportion to the distance through which the electron has fallen in the direction of E. This result could be, of course, anticipated since neither the magnetic field nor the elastic collisions contribute anything to the building up of the kinetic energy, and the latter is due exclusively to the electric field.

If the collisions occur at other points of the trochoidal trajectories (we still assume the central type of collisions for which $\phi = \pi$) the corresponding initial conditions must be taken into account in integrating equations (3). For example, a collision may occur at the lower part of the trochoid. For the lower apex it will be a $(\pi, 0)$ collision in our notation; it is easy to ascertain that its effect is opposite to that of a (π, π) collision.

The preceding results, found for a very special type of the "end on" collisions, can be easily generalized. One finds (in ξ, η, τ system):

$$\begin{aligned} \xi &= \tau - \rho \operatorname{Sin} \tau, \\ \eta &= -\rho \operatorname{Cos} \tau + \frac{\rho^2 + 1}{2}, \end{aligned} \tag{7}$$

where ρ is the generating radius.

Equations (7) represent a trochoid for a particular distance η_0 of the center of the trochoidal orbit from the cathode ($\eta_0 = \rho^2 + 1/2$). One verifies easily the results obtained in the particular case of the (π, π) collisions (table). Equation (7) is thus quite general, resulting from the geometry of the trochoidal motion as well as from the assumption of elastic collisions.

Let $\dot{\xi}_1 + i\dot{\eta}_1$ be the velocity of the electron at the instant immediately preceding a collision expressed as a complex quantity in the ξ, η plane, and let $\dot{\xi}_2 + i\dot{\eta}_2$ be that after the collision.

Since the collision is elastic immediately before and after the collision the velocity of the electron remains the same in magnitude but its direction changes abruptly by the angle ϕ. Hence

$$\begin{aligned}\dot{\xi}_2 + i\dot{\eta}_2 &= (\dot{\xi}_1 + i\dot{\eta})\, e^{i\phi} \\ &= (\dot{\xi}_1 \operatorname{Cos}\phi - \dot{\eta}_1 \operatorname{Sin}\phi) + i(\dot{\xi}_1 \operatorname{Sin}\phi + \dot{\eta}_1 \operatorname{Cos}\phi) \qquad (8)\end{aligned}$$

and

$$\dot{\xi}_1^2 + \dot{\eta}_1^2 = \dot{\xi}_2^2 + \dot{\eta}_2^2. \qquad (9)$$

Designating by τ_0 the instant of the collision, and noting that at the limit, $\xi_1 = \xi_0 = \xi_2$ and $\eta_1 = \eta_0 = \eta_2$, one has:

$$\begin{aligned}\xi_1 &= \tau_0 - \rho \operatorname{Sin}\tau_0 = \xi_2, \\ \xi_2 &= (1 - \rho \operatorname{Cos}\tau_0) \operatorname{Cos}\phi - \rho \operatorname{Sin}\tau_0 \operatorname{Sin}\phi, \\ \eta_1 &= -\rho \operatorname{Cos}\tau_0 = \eta_2, \\ \dot{\eta}_2 &= (1 - \rho \operatorname{Cos}\tau_0) \operatorname{Sin}\phi + \rho \operatorname{Sin}\tau_0 \operatorname{Cos}\phi.\end{aligned} \qquad (10)$$

The general integral of equations (3) is of the form:

$$\begin{aligned}\xi &= \pi + B \operatorname{Cos}\tau + C \operatorname{Sin}\tau + \tau, \\ \eta &= \quad - B \operatorname{Sin}\tau + C \operatorname{Cos}\tau + D,\end{aligned} \qquad (11)$$

where the constants of integration A, B, C, D are determined by the initial conditions (10). This finally gives the parameters of the new trajectory as a function of the corresponding parameters (ρ, η) of the old one and of the two additional parameters $(\tau_0\phi)$ determining the collision.

Omitting here the intermediate calculations one has finally:

$$\rho_2 = \sqrt{(\rho_1 \operatorname{Cos} \tau_0 - 1)^2 + \rho_1^2 \operatorname{Sin}^2 \tau_0 + 2\rho_1 \operatorname{Cos} (\tau_0 - \phi) - 2 \operatorname{Cos} \phi + 1}, \quad (12)$$

$$\eta_2 = D = \frac{\rho^2 + 1}{2} + 1 - \rho_1 \operatorname{Cos} \tau_0 + \rho_1 \operatorname{Sin} \tau_0 \operatorname{Sin} \phi + (\rho_1 \operatorname{Cos} \tau_0 - 1) \operatorname{Cos} \phi. \quad (13)$$

Equation (12) gives the generating radius ρ_2 of the new trajectory after the collision and (13) the ordinate η_2 of the center of rolling circle of this new trajectory, which is thus completely determined.

The preceding results for the particular case of the (π, π) collisions (table (6)) are easily found from the general formulæ (12) (13) by putting $\tau_0 = \pi$ and $\phi = \pi$. There exists a double infinity of new trajectories (ρ_2, η_2) corresponding to all possible values of the parameters τ_0, ϕ determining a collision.

For a given trajectory (ρ_1, η_1) the corresponding parameters ρ_2, η_2 of the new trajectory pass through maximum-minimum values when one has a straight "end on" apex collision (either (π, π) or $(\pi, 0)$ in our notation).[9] For these extreme values of scattering there exist simple relations

$$(\rho_2)\,\text{extr.} = \rho_1 \pm 2; \qquad (\eta_2)\,\text{extr.} = \frac{\rho_1^2 + 1}{2} + 2(1 \pm \rho_1) \quad (14)$$

From the second equation (14) it follows that the centers of new trochoidal trajectories scattered by collisions cannot leave a band of the width $4\rho_1$ around the line of center of the original trajectory. This gives the zone within which the scattering can spread around the initial trajectory.

The above elementary analysis shows that the electronic motions occuring in an inert gas under the specified conditions are characterized by a trochoidal field of electronic trajectories having that property that the rolling radius $a = E/kH^2$ and the orbital period $T = 2\pi/kH$ are constants of the field depending only on E and H.

Expressing the length and the time in a, τ units, the trochoidal field can be expressed in a single valued manner by means of equations (7).

[9] One finds this easily by equating to zero the partial derivatives of ρ_2 and η_2 with respect to the parameters τ_0, ϕ determining the collision and by solving the equations thus obtained.

DISTRIBUTION OF ELECTRONS IN A TROCHOIDAL FIELD.

The preceding study must be supplemented by statistical considerations in order to be able to establish the conditions of distribution of electrons on various virtual trochoidal trajectories which is equivalent to the formulation of the space charge conditions to the first approximation.

In order to simplify the study let us assume that instead of a double infinity of scattered trajectories depending on two independent parameters τ_0 and ϕ the scattering takes place by two discreet trajectories corresponding solely to either $(\pi\pi)$ or (πo) types of collisions.

Furthermore, in order to introduce the probability considerations it is *assumed that for a given virtual trajectory (equation (7)) the probability of a $(\pi\pi)$ collision is proportional to the length of the upper arc of the trajectory and that of a (πo) collision to the length of the lower arc of the same curve.*

This, in fact, is an a priori probability assumption which has no further justification,[10] and is equivalent to the *definition* of two statistical weights

$$\alpha_i = \frac{\sigma_i \text{ upper}}{\sigma 2\pi} \text{ and } \beta_i = \frac{\sigma_i \text{ lower}}{\sigma 2\pi}, \quad (\alpha_i + \beta_i = 1) \qquad (15)$$

to be attached to the ith trochoidal trajectory.

The calculation of the statistical weights α_i, β_i is thus reduced to that of the lengths of trochoidal tracks.

From the geometry of trochoidal curves one has:

$$d\sigma_i = \sqrt{1 + \rho_i^2 - 2\rho_i \operatorname{Cos} \tau}.d\tau$$

and

$$\sigma = (1 + \rho_i) \int_0^{\tau_1} \sqrt{1 - k_1^2 \operatorname{Cos}^2 \frac{\tau}{2}} \cdot d\tau$$

where

$$k_i^2 = \frac{4\rho_i}{(1 + \rho_i)^2}.$$

The upper limit τ_1 corresponds to the point of the trochoid for which $d\eta/d\xi = \infty$ that is for $\operatorname{Cos} \tau_1 = 1/\rho$.

[10] In case of a roulette having red sectors two times wider than the black ones and for equal number of sectors the a priori probability to gain on red is $\frac{2}{3}$ and on black $\frac{1}{3}$. We can designate this particular probability as probability of the phase to distinguish it from the probability of collision introduced in the next section.

By known transformation the above integral is reduced to the elliptic integral of the second kind. The values of α_i and β_i can be calculated in this manner.

It is possible now to apply the symbolic expansion of the product $\prod_{i=1}^{i=n} (\alpha_i S + \beta_i F)$ [11] in order to approximate the mechanism of electronic conduction in crossed fields.

In the above product S designates "success," that is, in this problem the passage of an electron from a lower to a higher trajectory by means of a $(\pi\pi)$ collision and F (failure) has a corresponding interpretation for a $(\pi 0)$ collision.

To complete our model of discreet trajectories it is preferable to consider that the electron emission occurs by abrupt steps separated by the periods $T = 2\pi/kH$ of no emission. Assume that at $\tau = 0$ the filament emits q electrons; they begin to describe the cycloidal trajectory (for which $\alpha_0 = 1$, $\beta_0 = 0$) the collisions occurring on this trajectory will cause a new re-distribution

$$\alpha_0 S \times (\alpha_1 S + \beta_1 F) = \alpha_0\alpha_1 S^2 + \alpha_0\beta_1 SF. \qquad (16)$$

It is necessary to agree as to the meaning of the symbolic product; [12] in a game of chance S^2 designates two subsequent successes, SF one success followed by one failure and so on. In this case there is no necessity of keeping S and F since the statistical weights $\alpha\beta$ already indicate the direction of scattering, namely, the statistical weight α indicates a $(\pi\pi)$ collision and β indicates a $(\pi 0)$ collision. Equation (16), therefore, can be written:

$$\alpha_0 \times (\alpha_1 + \beta_1) = \alpha_0\alpha_1 + \alpha_0\beta_1$$

which means that all the electrons α_0 transferred from the zero trajectory into the first, after the second collision undergo a new redistribution.

(1) $\alpha_0\alpha_1$ electrons go into the second trajectory ($\rho = 5$),
(2) $\alpha_0\beta_1$ electrons return back to the zero trajectory ($\rho = 1$),

the first group undergoes a new redistribution $\alpha_0\alpha_1(\alpha_2 + \beta_2) = \alpha_0\alpha_1\alpha_2 + \alpha_0\alpha_1\beta_1$ giving rise to $\alpha_0\alpha_1\alpha_2$ electrons on the third

[11] E. Borel, "Elements de la Theorie des Probabilites," Paris.
[12] E. Borel, "Theorie des Probabilites," page 15.

trajectory and $\alpha_0\alpha_1\beta_1$ electrons on the first ($\rho = 3$); the second group undergoes a redistribution $\alpha_0\beta_1(\alpha_0 + 0) = \alpha_0\beta_1\alpha_0$ which simply means that all these electrons are reflected back on the first trajectory ($\rho = 3$) since $\beta_0 = 0$.

Thus after three elastic collisions there will be

$\alpha_0\alpha_1\alpha_2$ electrons on the third trajectory ($\rho = 7$),
$\alpha_0\alpha_1\beta_1 + \alpha_0\beta_1\alpha_0$ on the first trajectory ($\rho = 3$),[13]

This step by step method can easily be applied to a gradually increasing number of collisions. The various products of statistical weights such as $\alpha_0\alpha_1\beta_1$ for example give the statistical weights of corresponding compound events. In the above example, that will mean the number of electrons which will first rise by two subsequent ($\pi\pi$) collisions from zero to the second trajectory and from there will descend by means of the third ($\pi 0$) collision to the first trajectory.

Proceeding in this manner gradually it is possible to represent graphically the probability curves attached to various levels. This is done on Fig. 1 for 28 subsequent collisions; the ordinates of the curves represent the sum of products of the statistical weights characterising the given trajectory plotted against the number of collisions; the figures attached to the curves designate electronic levels in which the distribution is represented by the probability curve. The abcissa axis does not represent any quantity proportional to the time in view of the fact that in each level there must be a factor taking into account the corresponding value of the mean free path; these factors are also different for various trajectories. This complication, however, is left out in this section [14] which is equivalent to the assumption that the collisions occur simultaneously in all levels.

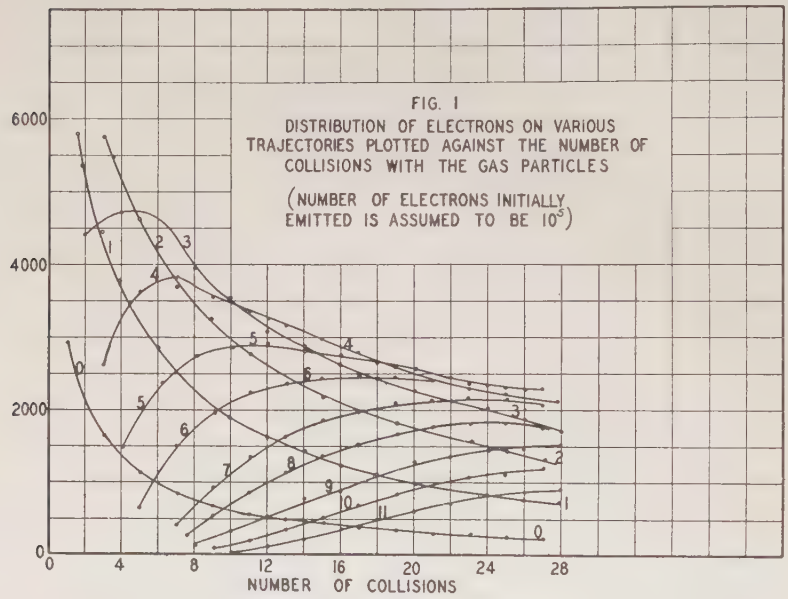
From the curves it is possible, at least qualitatively to form an idea as to how the initial layer of electrons emitted at $t = 0$ spreads itself through the gas. Near the filament the electrons drift away from the cathode comparatively swiftly since α weights are much greater than β weights for these trajectories.

[13] The condition that a trajectory of an even order gives rise to two trajectories of odd orders (and vice versa) results from the assumed discreet mechanism of collisions and has no further physical significance.

[14] See the following section.

With increasing number n of trajectories the difference $\alpha_n - \beta_n \underset{n}{\rightarrow} 0$ and this manifests itself statistically by a flattening out and spreading of the probability curves along the abscissa axis. The electrons have a tendency to be delayed in the gas without going promptly to the anode and the existence

FIG. 1.



of a negative space charge in the gas can be understood on this basis.

To complete the picture one should superimpose on the distribution given by curves subsequent distributions corresponding to the emission at the subsequent instants.

SPACE CHARGE.

The preceding picture is not complete in that it considers only the probability of a given phase on trochoids at which the collisions occur. In addition to this *probability of phase* it is necessary to introduce also the *probability of collision* depending on the mean free path.

Let x_i be the number of electrons lost from the ith trajectory per unit time by the collisions with the gas particles. With the assumed mechanism of collisions $x_i\alpha_i$ and $x_i\beta_i$ are cor-

responding numbers of electrons reflected to the nearest two trajectories on the anode and the cathode sides respectively. The probability for a group of electrons q of travelling without collision through a distance σ along the cycloidal track is $e^{-\sigma/\lambda}$ where λ is the mean free path. During the orbital period $\tau = 1$ this trajectory will receive $q + x_1\beta_1$ electrons (q electrons from the filament and $x_1\beta_1$ from the first trajectory) and $(q + x_1\beta_1)e^{-\sigma_0/\lambda}$ is the number of electrons left on the zero trajectory after the distance σ_0. But in addition to these electrons the same trajectory may be still occupied by the electrons emitted prior to the instant of the last emission.

Hence the total number of electrons present on the zero trajectory is $X_0 = \sum_{n=0}^{n=\infty} (n)(q + x_1\beta_1)e^{-\sigma_0/\lambda}$.

Using integral notation for the limit.

$$X_0 = (q + x_1\beta_1)\int_0^\infty e^{-\sigma_0/\lambda}d\tau, \tag{17}$$

where

$$\sigma = \int_0^\tau \sqrt{1 - 2\rho\cos\tau + \rho^2}\cdot d\tau$$

and

$$\sigma_0 = \int_0^\tau \sqrt{2(1 - \cos\tau)}d\tau.$$

For larger values of ρ, σ can be approximated by $\rho\tau$, that is a circle substituted for the trochoid. In this case $X_0 = (q + x_1\beta_1)\,\lambda/\rho$. It is convenient to use the approximation $\sigma = \rho\tau + \epsilon(\rho)$ even for the trajectories of lower order ($\rho = 1, 3,$) by introducing a suitable function $|\epsilon(\rho)| \underset{\rho\to\infty}{\rightrightarrows} 0$. The calculation shows that $\epsilon(\rho)$ approaches the zero very rapidly with increasing ρ and already for the third trajectory it is less than 1 per cent. of $\rho\tau$.

The average electron densities in various belts are

$$\begin{aligned}
\Delta_0 &= \frac{X_0}{2\rho_0} = (q + x_1\beta_1)\frac{\lambda}{2\rho_0^2},\\
\Delta_1 &= \frac{X_1}{2\rho_1} = (x_0\alpha_0 + x_2\beta_2)\frac{\lambda}{2\rho_1^2},\\
\Delta_n &= \frac{X_n}{2\rho_n} = (x_{n-1}\alpha_{n-1} + x_{n+1}\beta_{n+1})\frac{\lambda}{2\rho_n^2}.
\end{aligned} \tag{18}$$

In the above expressions $x_0 \ldots x_n \ldots$ are unknown. Their determination, however, is immediate in case of a steady state. Consider the nth trajectory: it loses $x_n\alpha_n$ electrons towards the anode and $x_n\beta_n$ towards the cathode since $\alpha_n > \beta_n$ the trajectory loses more electrons towards the anode and the difference $x_n(\alpha_n - \beta_n) = q$ is the amount of electricity transferred through the gas per unit time. Clearly

$$x_1(\alpha_1 - \beta_1) = x_2(\alpha_2 - \beta_2) = \cdots x_n(\alpha_n - \beta_n) = q.$$

Since $\alpha_n + \beta_n = 1$ for any trajectory one has

$$x_1 = \frac{q}{2\alpha_1 - 1} \cdots x_n = \frac{q}{2\alpha_n - 1}.$$

For increasing n the coefficients α_n approach assymptotically the value 0.5, that is x_n increases indefinitely. Thus the trajectories with greater generating radii lose more electrons than those of smaller radii which is physically obvious since in the former case the electrons are travelling longer paths in the gas than in the latter.

The average electron density in the nth belt is given by

$$\Delta_n = \left(1 + \frac{1 - \alpha_{n+1}}{2\alpha_{n+1} - 1}\right)\frac{q\lambda}{\rho_n^2} = f_n \frac{q\lambda}{\rho_n^2}; \qquad (19)$$

it is represented graphically on fig 2; it is seen that the function Δ_n is decreasing when one moves away from the cathode. The zone of the greatest electron concentration is very near to the cathode where the trajectories have rather small generating radii and therefore a rather small probability of collision with the gas particles.

The electric force in a gas is given by the first integration of the one dimensional Poisson equation in which η is taken as independent variable instead of ρ.

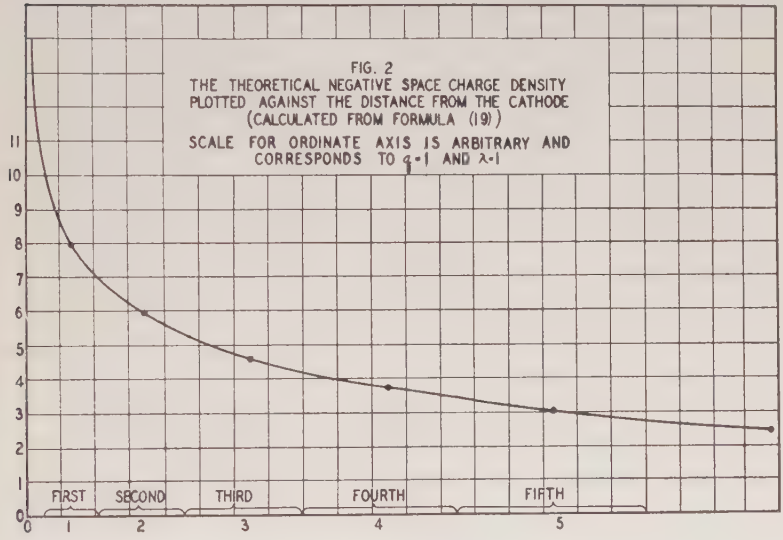
$$\frac{dV}{d\eta} = -4\pi \int_0^\eta \Delta(\eta)d\eta + \left(\frac{dV}{d\eta}\right)_0$$

and the potential

$$V = -4\pi \int_0^\eta d\eta \int_0^\eta \Delta(\eta)d\eta + \int_0^\eta \left(\frac{dV}{d\eta_0}\right)d\eta + V_0, \qquad (20)$$

where $dV/d\eta_0$ and V_0 are two constants of integration. One can also determine the potential difference across a gas graphically by double integration of the curve between o and a certain ρ_n. When the magnetic field is increased the size of trochoidal loops decreases so that there are more trochoidal trajectories [15] between the anode and the cathode—this is

FIG. 2.



equivalent to the increased value of the upper limit of integration which increases the value of the integral, since the integrand is an essentially positive quantity.

When the mean free path is increased, the number of trajectories between the cathode and the anode remains the same but the density in each level increases, which causes again an increase of the potential difference in the gas.

In summing up the potential difference across the gas in which the trochoidal field of electronic trajectories is established depends on two independent parameters H and λ.[16]

[15] This of course relates to the discreet trajectories (without over-lapping) as is assumed in this section.

[16] See in this connection the earlier paper of the writer in Journal de Physique (April, 1928, Section 7).

With the assumed mechanism of collisions, the action of H analytically corresponds to the variation of the upper limit of integration of the one dimensional Poisson's equation applied to the trochoidal field and the action of λ to that of an independent parameter of the integrand.

IONIZATION IN CROSSED FIELDS.

The preceding investigation can be extended to take into account the case of non elastic impacts. This corresponds to the case when the linear velocity (expressed in volts) on a trajectory becomes at least equal to, or greater than, the first critical potential of the gas. For a sufficiently high value of this orbital velocity the ionization potential may be reached and there appears a probability of ionization in the gas.[17]

In an elementary act of ionization a quantum of energy is subtracted from the kinetic energy of the electron and transferred into the atom. At this point the trochoidal trajectory undergoes a discontinuity shrinking from a trochoid with a comparatively large generating radius down to the cycloid in case the electron loses all its velocity or, at least to a trochoid of smaller generating radius.

The zone of ionization in the gas, consequently, will be characterized by trochoidal trajectories of much smaller generating radii appearing in this zone. This is also applicable to the secondary electrons released from the atoms during the process of ionization.[18]

The trochoidal field under these circumstances loses its single valued determination by equations (7). To a given level, η does not correspond a definite ρ as was the case in elastic collisions.

The model of the trochoidal field, although considerably limited in case of ionization can be still used to explain the

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[17] R. Seeliger, *Ann. d. Phys.*, Bd. 59, 1919, *Zs. f. Phys.*, Bd. 2, 1920, Bd. 4, 1921, Bd. 11, 1922, Bd. 15, 1923; Seeliger, R., und Mierdel, G., *Zs. f. Phys.*, Bd. 1, 1920, Bd. 5, 1921.

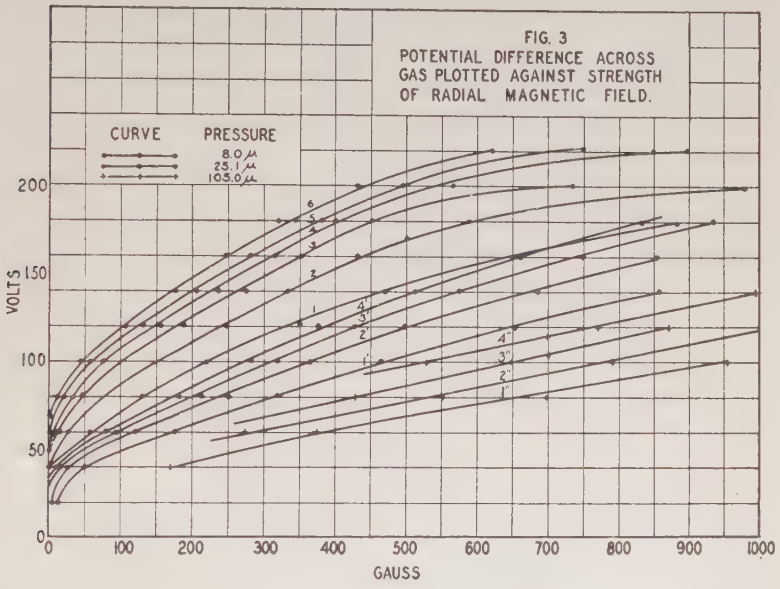
[18] The presence of positive ions modifies the space charge distribution. It is reasonable to expect, however, that this influence is not so important in the present case. In fact the positive ions owing to their small value of e/m are hardly influenced by magnetic field and are not much delayed in the gas. The presence of very high potential gradients in the gas seems to corroborate also this conclusion.

existence of high potential gradients in an ionized gas placed in a transverse magnetic field.

In case of an arc placed in a radial magnetic field the existence of high potential gradients depending on two parameters H and λ was actually observed prior to the formulation of the above theory.

In case of less intense ionization in argon at comparatively low pressures[19] a similar condition exists. This is evidenced for example by the curves of Fig. 3 taken for variable H and three

FIG. 3.

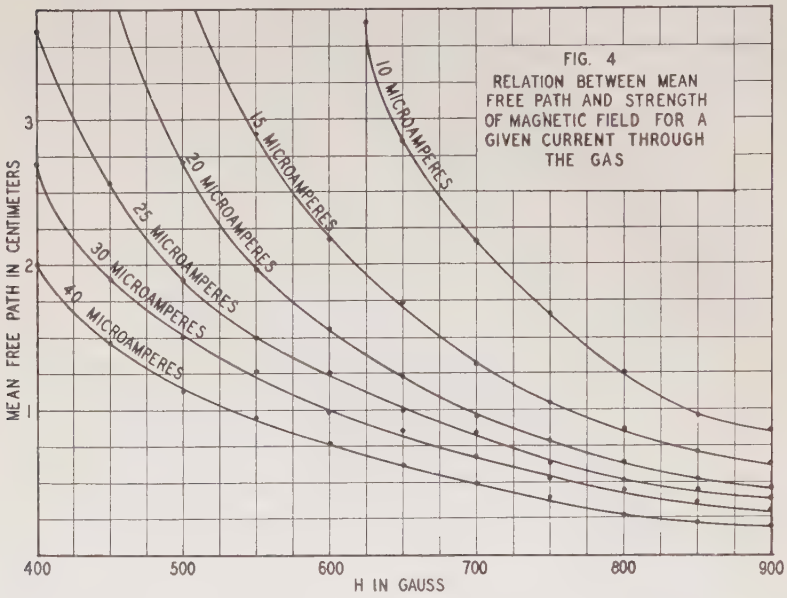


different values of λ corresponding to the argon pressures of 8, 25 and 105μ respectively. Curves 1, 2, 3, 4 etc. were taken for the currents of .1, .2, .3, etc. milliamps respectively. It is seen that for 105μ the voltage across the gas increases approximately in proportion to H.

[19] These experiments will be described in a later publication; it is sufficient to mention here that in these experiments a circular tungsten filament was used instead of mercury in the same tube with which the arc experiments were previously made (N. Minorsky, *loc. cit.*).

The curves of Fig. 4 give the relation between H and λ for definite currents through the gas. It is seen that for increasing H the mean free path decreases which is also apparent on the basis of the above theory. In fact, for the increasing H the rolling radius $a = E/e/mH^2$ decreases for the

FIG. 4.



whole trochoidal field which causes the decrease, on the average of the difference $(\alpha_n - \beta_n)$ on all trajectories between the cathode and the anode and this corresponds to a smaller probability of transition toward the anode. If, however, the current through the gas must remain the same the only way to accomplish this is to act on the remaining degree of freedom viz., to increase the probability of collision by reducing the value of λ.

For a given mean free path the ionization current is the greater the stronger the field H is; this increase in ionization current apparently is not due to the probability of transition on trochoidal trajectories since the magnetic field in case of elastic impacts decreases this probability of transition. One must think that this phenomenon is due rather to the increased

probability of collision which, above the ionization potential, corresponds also to the increased probability of ionization in the gas. The increase of ionization current with H is much more considerable for large values of λ than for the small ones. This seems to be also the case for the potential gradient in the gas which is much more influenced by H in case λ is large. It is very easy, for example, to raise the voltage across an arc placed in a radial field say two–three times its initial value (for $H = 0$) when the temperature is low and λ great but, as soon as λ is reduced, even very large changes in H hardly cause any change at all on the voltage across the arc. All phenomena relative to conduction and especially to ionization in crossed fields acquire particularly interesting character for comparatively low pressures.

On the basis of the above theory it seems possible to explain this feature by a statistical process of collisions occurring in a trochoidal field between the electrons and the gas particles governed in the first approximation by two independent probabilities viz., probability of the phase depending on the magnetic field H and probability of collision depending both on H and λ, to which in case of ionization adds up also the probability of ionization, depending presumably on the orbital velocity of electrons.